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Internal waves in a continuously stratified atmosphere or ocean

By JOHN W. MILES

Institute of Geophysics and Planetary Physics, University of California, La Jolla, California†

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A stably stratified shear flow has the velocity profile $V(1-e^{-y/h})$ and the density profile $\log(\rho_0/\rho) = \sigma(1-e^{-y/h})$ in $0 < y < \infty$ and is bounded by the rigid plane $y = 0$. It is proved that small disturbances with respect to this basic flow are stable for all wavelengths and Richardson numbers. The eigenvalues for neutral disturbances (internal gravity waves) are enumerated. The results are applicable to the atmosphere and to an infinitely deep ocean.

1. Introduction

We consider the heterogeneous shear flow described by the velocity profile

$$U(y) = V(1-e^{-y/h}) \tag{1.1}$$

and the density profile

$$\lambda(y) = \log[\rho_0/\rho(y)] = \sigma(1-e^{-y/h}) \quad (0 < \sigma \ll 1) \tag{1.2}$$

in an inviscid, incompressible fluid that is bounded by a rigid horizontal plane, $y = 0$, and fills the half-space $y > 0$; x and y are Cartesian co-ordinates. Describing travelling-wave disturbances, relative to this basic flow, by the perturbation stream function

$$\psi = Re\{\phi(y)\exp[ik(x-ct)]\} \quad (0 < y < \infty), \tag{1.3}$$

we seek an enumeration of the admissible values of the wave speed c.

It would be desirable to obtain this enumeration for arbitrary prescribed values of the dimensionless wave and Richardson numbers, say

$$\alpha = kh, \quad J = \sigma gh/V^2. \tag{1.4a, b}$$

In fact, we find it necessary to give the enumeration for a prescribed value of the local Richardson number at $y = y_c$, where $U(y_c) = c$. The local Richardson numbers for arbitrary y and $y = y_c$ are given by

$$J(y) = g\lambda'(y)/U'^2(y) = J\,e^{y/h}, \quad J(y_c) = JV/(V-c) \equiv Jw_0, \tag{1.5a, b}$$

where the introduction of w_0 anticipates (2.3c) below.

The only heterogeneous shear flow over a rigid boundary for which the admissible c have been enumerated is that described by a linear velocity profile and an exponential density profile, say $U = Vy/h$ and $\lambda = \sigma y/h$; these profiles can be

† Also Department of the Aerospace and Mechanical Engineering Sciences.

obtained from (1.1) and (1.2) by letting $h \to \infty$ with V/h and σ/h fixed. This configuration was considered originally by Taylor (1931), who demonstrated that no waves exist for $0 < J < \frac{1}{4}$ and that an infinite number of neutral, retrograde waves exists for $J > \frac{1}{4}$. Taylor conjectured, but was unable to demonstrate, that no unstable waves exist for $J > \frac{1}{4}$; Eliassen, Høiland & Riis (1953) gave the required proof. See also Case (1960) and Dyson (1960).

Taylor's configuration, regarded as a model of a shear layer with a scale small compared with the height of the atmosphere, has the disadvantage of unbounded $U(y)$ and $\lambda(y)$. It seems natural to consider the effects of requiring $U(y)$ and $\lambda(y)$ to be bounded and, in particular, to inquire whether there exist neutral waves with phase speeds in excess of the maximum wind speed (we anticipate that such waves do exist). A consideration of the differential equation for $\phi(y)$, (2.1) below, with special reference to its singularities (see Miles 1963 for a detailed discussion), reveals that the heterogeneous shear flow described by (1.1) and (1.2) is the simplest possible under the restrictions that $U(y)$ and $\lambda(y)$ be both bounded and continuous. The resulting differential equation has three regular singularities, and the eigenvalue problem reduces to the determination of the zeros of the hypergeometric function. We emphasize, however, that (1.1) implies $U''(y) < 0$ everywhere, in consequence of which our conclusions are not likely to be significant for velocity profiles that possess flex points, which typically imply instability for $J = 0$. The homogeneous ($J = 0$) shear flow described by (1.1) corresponds to the asymptotic form of a boundary layer with suction and is stable for all α (Hughes & Reid 1965).

We have posed the velocity and density profiles of (1.1) and (1.2) in an essentially atmospheric context (U increasing and ρ decreasing with y). We obtain the corresponding oceanographic problem if we replace (1.1)–(1.3) by

$$\psi = Re\{\phi(y)\exp(ik[x-(V-c)t])\} \quad (-\infty < y < 0), \tag{1.6}$$

$$U(y) = V e^{y/h} \tag{1.7}$$

and

$$\lambda(y) = \sigma(e^{y/h} - 1). \tag{1.8}$$

The boundary condition of vanishing vertical displacement at $y = 0$, as posed in (2.2a) below, is appropriate for internal waves in a stratified ocean provided that the characteristic speed V is small compared with the speed of surface waves; this requires

$$kV^2/g = (\alpha/J)\sigma \ll 1, \tag{1.9}$$

which is satisfied by typical ocean currents.

2. Formulation of the eigenvalue problem

The restriction $\sigma \ll 1$ permits the usual Boussinesq approximation, whereby the density is regarded as constant except in the calculation of the specific buoyancy force, $g\lambda'(y)$. The resulting differential equation for $\phi(y)$ is

$$\phi'' + [g\lambda'(U-c)^{-2} - U''(U-c)^{-1} - k^2]\phi = 0. \tag{2.1}$$

Invoking the requirements that the vertical displacement vanish at the lower boundary and that the solution be bounded at infinity, we obtain the boundary conditions

$$(U-c)^{-1}\phi = 0 \quad (y = 0), \quad \phi = 0 \quad (y = \infty) \qquad (2.2a, b)$$

which, together with (2.1), define the eigenvalue problem. See Miles (1961, 1963) for a more complete discussion of this formulation.

Introducing the transformation†

$$\phi(y) = e^{-ky}f(w), \quad w = w_0 e^{-y/h}, \quad w_0 = V/(V-c), \qquad (2.3a, b, c)$$

we find that $f(w)$ satisfies Riemann's differential equation

$$f = P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \alpha-(1+\alpha^2)^{\frac{1}{2}} & \frac{1}{2}(1+\nu) & w \\ -2\alpha & \alpha+(1+\alpha^2)^{\frac{1}{2}} & \frac{1}{2}(1-\nu) & \end{Bmatrix}, \qquad (2.4)$$

where $$\nu = (1-4Jw_0)^{\frac{1}{2}} \equiv i\mu. \qquad (2.5)$$

Invoking (2.2*b*), which implies that $f(w)$ must be regular at $w = 0$, we choose

$$f(w) = (1-w)^{\frac{1}{2}(1+\nu)}F(a, b; 1+2\alpha; w) \equiv (1-w)^{\frac{1}{2}(1+\nu)}F(w), \qquad (2.6)$$

where $$a,\, b = \tfrac{1}{2}(1+\nu)+\alpha \mp (1+\alpha^2)^{\frac{1}{2}}, \qquad (2.7)$$

$F(w)$ is the hypergeometric series in a w-plane cut along $(1, \infty)$, and we omit an arbitrary constant multiplier. Invoking the transformation formulas for $F(w)$, we obtain the alternative representations

$$f(w) = (1-w)^{\frac{1}{2}(1-\nu)}F(a^*, b^*; 1+2\alpha; w) \equiv (1-w)^{\frac{1}{2}(1-\nu)}F^*(w) \qquad (2.8)$$

and $$f(w) = A^*(1-w)^{\frac{1}{2}(1+\nu)}F(a, b; 1+\nu; 1-w) + A(1-w)^{\frac{1}{2}(1-\nu)}F(a^*, b^*; 1-\nu; 1-w), \qquad (2.9)$$

where $$A = \Gamma(1+2\alpha)\Gamma(\nu)/\Gamma(a)\Gamma(b), \qquad (2.10)$$

and the asterisk implies the transformation $\nu \to -\nu$ (but F^* is the complex conjugate of F if and only if w_0 is positive-real and $J_c > \frac{1}{4}$).

Invoking the boundary condition (2.2*a*), we obtain the eigenvalue equation

$$(1-w_0)^{\frac{1}{2}(-1+\nu)}F(w_0) = 0. \qquad (2.11)$$

3. Distribution of eigenvalues

We require the roots of (2.11) or, equivalently, the zeros of the hypergeometric function $F(w_0)$ in a plane cut along $w_0 = (1, \infty)$. The transformation (2.3*c*) implies a one-to-one mapping of the c-plane on the w_0-plane, with real c mapping on to real w_0 according to $c = (-\infty, 0, V, \infty) \to w_0 = (0+, 1, \infty, 0-)$; see table 1.

It evidently is expedient to describe the distribution of the zeros in terms of the parameters α and ν, rather than α and J. Assuming first that ν is real ($Jw_0 < \frac{1}{4}$), so that each of the hypergeometric parameters a, b, and $1+2\alpha$ also is real, we find (see Van Vleck 1902) that $F(w_0)$ has (i) one zero in $w_0 = (0, 1)$ if and only if $0 < \nu < \nu_0$; (ii) no zeros in $w_0 = (1, \infty)$; and (iii) n zeros in $w_0 = (-\infty, 0)$ if $\nu_n < \nu < \nu_{n+1}$, where

$$\nu_0(\alpha) = 2(1+\alpha^2)^{\frac{1}{2}} - 2\alpha - 1 \quad (\alpha \leqslant \tfrac{3}{4}) \qquad (3.1a)$$

$$= 0 \quad (\alpha \geqslant \tfrac{3}{4}) \qquad (3.1b)$$

and $$\nu_n(\alpha) = 2[(1+\alpha^2)^{\frac{1}{2}}+\alpha]+2n+1. \qquad (3.2)$$

These results are tabulated in rows (i*a*), (ii), (iii) of table 1.

† The sign of y must be changed in the exponents of (2.3*a*, *b*) if (1.1)–(1.3) are replaced by (1.6)–(1.8).

A new zero of $F(w_0)$ must originate at one of the two singular points, $w_0 = 1$ $(c = 0)$ or $w_0 = \infty\,(c = V)$. In particular, $F(w_0)$ gains a zero at $w_0 = 1-$ as ν decreases through $\nu_0(\alpha)$ and at $w_0 = -\infty$ as ν increases through $\nu_n(\alpha)$. The former transition is governed by the analytical continuation (2.9), from which we infer that

$$c/V \to -[\tfrac{1}{2}(\nu_0-\nu)\,B(1+2\alpha,\nu_0)]^{1/\nu_0}, \quad (1-4J)^{\frac{1}{2}} \to \nu_0(\alpha)- \qquad (3.3a)$$

$$\doteqdot -(J-\alpha), \quad (0 < \alpha < J \ll 1). \qquad (3.3b)$$

	c	w_0	ν	N
(ia)	$(-\infty, 0)$	$(0, 1)$	$(0, \nu_0)$	1
(ib)			$(0, i\infty)$	∞
(ii)	$(0, V)$	$(1, \infty)$	$(0, 1)$	0
(iii)	(V, ∞)	$(-\infty, 0)$	(ν_n, ν_{n+1})	n

TABLE 1. The distribution of the eigenvalues. N is the number of eigenvalues in the indicated range of c

Similar results, governing the appearance of a new eigenvalue at $w_0 = -\infty$ as ν increases through $\nu_n(\alpha)$, can be obtained from the analytical continuation of $F(w_0)$ into the neighbourhood of $w_0 = -\infty$ with $b^* \to -n$. We remark that there is an infinite number of such zeros, with

$$-Jw_0 \to \tfrac{1}{4}(\nu_n^2-1), \quad w_0 \to -\infty, \quad J \to 0+ \quad (n = 1, 2, \ldots), \qquad (3.4)$$

in consequence of which $c = V+$ is a limit point for the eigenvalues in $c > V$, and the limiting case of a homogeneous shear flow $(J = 0)$ is degenerate.

A necessary condition for complex c is the existence of singular neutral modes, for which $0 < c < V\,(1 < w_0 < \infty)$, along some locus in an (α, ν)-plane; singular neutral modes cannot exist for imaginary values of ν (Miles 1961). Having demonstrated that $F(w_0)$ has no zeros in $w_0 = (1, \infty)$ for real values of ν, we infer that there are no unstable modes for any α and J.

Now let $Jw_0 > \frac{1}{4}$, so that ν is imaginary. Setting $\nu = i\mu$ in (2.9) and (2.10), we rewrite (2.11) in the form

$$Re\{A^*(1-w_0)^{\frac{1}{2}(-1+i\mu)}F(a, b;\, 1+i\mu;\, 1-w_0)\} = 0, \qquad (3.5)$$

where the asterisk now implies complex conjugation. Following the proofs given by Taylor (1931) and Dyson (1960) for the corresponding equations involving a Hankel function and a confluent hypergeometric function, respectively, we can prove that (3.5) has an infinite number of zeros in $w_0 = (0, 1)$ and that these zeros have a limit point at $w_0 = 1-(c = 0-)$. There are no other zeros since, by hypothesis, $Jw_0 > 0$, and $w_0 = (1, \infty)$ is excluded for imaginary values of ν.

We designate the zeros in $w_0 = (0, 1)$ for imaginary values of ν by (ib); see table 1. The transition from (ia) to (ib), wherein the infinite set of eigenvalues in $Jw_0 > \frac{1}{4}$ goes over to a set of either one or zero eigenvalues in $0 < Jw_0 < \frac{1}{4}$, must occur at the singular point $w_0 = 1$. Letting $\nu \to 0$ and $w_0 \to 1-$ in (2.10) and (3.5), we obtain the limiting eigenvalue equation

$$-(J-\tfrac{1}{4})^{\frac{1}{2}}\cot[(J-\tfrac{1}{4})^{\frac{1}{2}}\log(1-w_0)] = \tfrac{1}{2}\nu_0(\alpha) \quad (J \to \tfrac{1}{4},\ w_0 \to 1-), \qquad (3.6)$$

where $\nu_0(\alpha)$ is now given by (3.1*a*) for all $\alpha > 0$. The left-hand side of (3.6) takes any value in $(-\infty, \infty)$ an infinite number of times as $w_0 \to 1-$ with $J > \frac{1}{4}$ and tends to $(\frac{1}{4}-J)^{\frac{1}{2}} < \frac{1}{2}$ as $w_0 \to 1-$ with $J < \frac{1}{4}$. The right-hand side decreases monotonically from $\frac{1}{2}$ to 0 as α increases from 0 to $\frac{3}{4}$ and is negative for $\alpha > \frac{3}{4}$. It follows that (3.6) has an infinite number of roots with a limit point at $w_0 = 1-$ for $J > \frac{1}{4}$ and any α and either one or no roots for $J < \frac{1}{4}$ and $\alpha < \frac{3}{4}$ or $\alpha > \frac{3}{4}$, respectively.

4. Asymptotic solutions

We consider briefly the limits $\alpha \to \infty$ and $J \to \infty$.

The most direct approach for $\alpha \to \infty$ appears to follow from the introduction of

$$z = \alpha(1-w) = \alpha(U-c)/(V-c) \tag{4.1}$$

in place of w in (2.3). We then obtain

$$\phi \sim z^{\frac{1}{2}} e^{-ky+z} K_{\frac{1}{2}\nu}(z)\,[1+O(1/\alpha)] \quad (\alpha \to \infty), \tag{4.2}$$

where $K_{\frac{1}{2}\nu}$ is a modified Bessel function of the second kind. The corresponding eigenvalues are the roots of

$$K_{\frac{1}{2}\nu}[-\alpha c/(V-c)] = 0 \quad (\alpha \to \infty) \tag{4.3}$$

and, as might have been anticipated, are identical with these for Taylor's configuration and belong to (i*b*) of table 1.

Turning to the limit $J \to \infty$, we find that the hypergeometric series of (2.7) has the asymptotic approximation

$$F(w) \sim \Gamma(1+2\alpha)\,(Jw_0 w)^{-\alpha} J_{2\alpha}\{2(Jw_0 w)^{\frac{1}{2}}\}\{1+O(Jw_0)^{-1}\} \quad (Jw_0 \to \infty). \tag{4.4}$$

The corresponding zeros of $F(w_0)$ belong to (i*b*) of table 1 and are given by

$$w_0 = \tfrac{1}{2} J^{-\frac{1}{2}} x_n(\alpha) \quad (J \to \infty), \tag{4.5}$$

where $x_n(\alpha)$ $(n = 1, 2, \ldots)$ are the positive zeros of $J_{2\alpha}(x)$. Abandoning our dimensionless notation, we obtain

$$c = V - 2(\sigma g h)^{\frac{1}{2}} x_n^{-1}(\alpha) \quad (V^2/\sigma g h \to 0), \tag{4.6}$$

which contains the limiting result for gravity waves in a stratified fluid without shear $(V = 0)$.

5. Conclusions

We conclude that the heterogeneous shear flow described by (1.1) and (1.2) is stable with respect to small disturbances of all wavelengths and will support an infinite number of internal gravity waves with wave speeds that lie outside of the range of wind speeds, $(0, V)$, but have limit points at $c = 0-$ for $J > \frac{1}{4}$ and at $c = V+$ for $J = 0+$.

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